Central extensions of restricted Lie superalgebras and classification of p-nilpotent Lie superalgebras in dimension 4

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Non-associative Algebras, Representations, and Applications

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1. Preliminaries

2. Restricted cohomology and central extensions
   - A (very) brief history of restricted cohomology
   - Restricted cohomology for restricted Lie superalgebras
   - Central extensions of restricted Lie superalgebras

3. Classification of low dimensional restricted Lie superalgebras
   - A brief history of classification of restricted Lie algebras
   - Dimension 3
   - Dimension 4

Restricted Lie algebras

Definition (Jacobson)

A **restricted Lie algebra** *is a Lie algebra L equipped with a map $(\cdot)^{[p]} : L \longrightarrow L$ satisfying for all $x, y \in L$ and for all $\lambda \in \mathbb{K}$:*

1. $(\lambda x)^{[p]} = \lambda^p x^{[p]}$;
2. $\left[x, y^{[p]}\right] = [[\cdots[x, \overbrace{y], y], \cdots, y}^{p \text{ terms}}]$;
3. $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$,

Nathan Jacobson (1910-1999)

with $is_i(x, y)$ the coefficient of Z^{i-1} in $\mathrm{ad}^{p-1}_{Zx+y}(x)$. Such a map $(-)^{[p]} : L \longrightarrow L$ is called p-map.

Example: any associative algebra A with $[a, b] = ab - ba$ and $a^{[p]} = a^p$, $\forall a, b \in A$.

Restricted Lie algebras

Definition

A Lie algebra morphism $f : \left(L, [\cdot,\cdot], (\cdot)^{[p]}\right) \to \left(L', [\cdot,\cdot]', (\cdot)^{[p]'}\right)$ *is called* **restricted** *if*

$$f\left(x^{[p]}\right) = f(x)^{[p]'}, \ \forall x \in L.$$

A L-module M is called **restricted** *if*

$$x^{[p]} \cdot m = \Big(\overbrace{x \cdot (x \cdots (x}^{p \text{ terms}} \cdot m) \cdots)\Big), \ \forall x \in L, \ \forall m \in M.$$

Restricted Lie superalgebras

Definition (Restricted Lie superalgebra)

A **restricted Lie superalgebra** *is a Lie superalgebra* $L = L_{\bar{0}} \oplus L_{\bar{1}}$ *such that*

1. *The even part* $L_{\bar{0}}$ *is a restricted Lie algebra;*
2. *The odd part* $L_{\bar{1}}$ *is a Lie* $L_{\bar{0}}$*-module;*
3. $[x, y^{[p]}] = [[\ldots[x, \overbrace{y], y], \ldots, y}^{p \text{ terms}}], \ \forall x \in L_{\bar{1}}, \ y \in L_{\bar{0}}.$

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We can define a map $(\cdot)^{[2p]} : L_{\bar{1}} \to L_{\bar{0}}$ *by*

$$x^{[2p]} = \left(x^2\right)^{[p]}, \ \text{with } x^2 = \frac{1}{2}[x, x], \ x \in L_{\bar{1}}.$$

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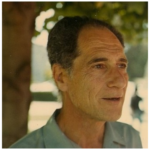
Theorem (Jacobson)

Let $(e_j)_{j \in J}$ *be a basis of* $L_{\bar{0}}$*, and let the elements* $f_j \in L_{\bar{0}}$ *be such that* $(\mathrm{ad}_{e_j})^p = \mathrm{ad}_{f_j}$*. Then, there exists exactly one* $p|2p$*-mapping* $(\cdot)^{[p|2p]} : L \to L$ *such that*

$$e_j^{[p]} = f_j \quad \text{for all } j \in J.$$

A (very) brief history of restricted cohomology

- 1955 (Hochschild): $H^n_*(L, M) := \mathrm{Ext}^n_{U_p(L)}(\mathbb{F}, M)$.



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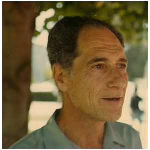

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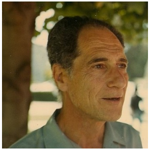
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- 2020 (Yuan-Chen-Cao): attempt to generalize to the superalgebras case.

Restricted cohomology for restricted Lie superalgebras

Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a restricted Lie superalgebra and let M be a L-module.

We set $C^0_*(L, M) = M$ and $C^1_*(L, M) = \mathrm{Hom}(L, M)$.

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Definition (Restricted 2-cochains)

*Let $\varphi \in C^2_{CE}(L, M)$ (ordinary Chevalley-Eilenberg 2-cochain) and $\omega : L \longrightarrow M$. Then ω is φ-**compatible** if*

1. $\omega(\lambda x) = \lambda^p \omega(x), \ \forall \lambda \in \mathbb{F}, \ \forall x \in L_{\bar{0}}$;

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2. $\omega(x+y) = \omega(x) + \omega(y) +$

$$\sum_{\substack{x_i = x \text{ or } y \\ x_1 = x, \ x_2 = y}} \frac{1}{\sharp(x)} \sum_{k=0}^{p-2} (-1)^k x_p \cdots x_{p-k+1} \varphi\Big([[\cdots[x_1, x_2], x_3] \cdots, x_{p-k-1}], x_{p-k}\Big),$$

with $x, y \in L_{\bar{0}}$, $\sharp(x)$ the number of factors x_i equal to x.

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with $x, y \in L_{\bar{0}}$, $\sharp(x)$ the number of factors x_i equal to x.

$$C^2_*(L, M) := \big\{(\varphi, \omega), \ \varphi \in C^2_{CE}(L, M), \ \omega \text{ is } \varphi\text{-compatible}\big\}$$

$\rightsquigarrow$ We have a similar (although more complicated) definition for $C^3_*(L, M)$.

Restricted cohomology for restricted Lie superalgebras

For $(\varphi, \omega) \in C^2_*(L; M)$, we write

$$(\varphi, \omega) = (\varphi_{\bar{0}}, \omega_{\bar{0}}) + (\varphi_{\bar{1}}, \omega_{\bar{1}}), \text{ where } \operatorname{Im}(\omega_{\bar{j}}) \subseteq M_{\bar{j}}. \tag{1}$$

Observe that also $(\varphi_{\bar{j}}, \omega_{\bar{j}}) \in C^2_*(L; M)$, thanks to the φ-compatibility.

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In the sequel we will define the maps

$$0 \longrightarrow \underset{(=C^0_{\mathrm{CE}}(L,M))}{C^0_*(L, M)} \xrightarrow{d^0_*} \underset{(=C^1_{\mathrm{CE}}(L,M))}{C^1_*(L, M)} \xrightarrow{d^1_*} C^2_*(L, M) \xrightarrow{d^2_*} C^3_*(L, M).$$

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First, we take $d^0_* := d^0_{CE}$.

Restricted cohomology for restricted Lie superalgebras

Definition of the map $d^1_* : C^1_*(L, M) \longrightarrow C^2_*(L, M)$.

An element $\varphi \in C^1_*(L; M)$ induces a map $\mathrm{ind}^1(\varphi) : L_{\bar{0}} \to M$ given by

$$\mathrm{ind}^1(\varphi)(x) = \varphi\big(x^{[p]}\big) - x^{p-1}\varphi(x).$$

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Theorem (Evans-Fuchs)

1. *The map* $ind^1(\varphi)$ *is* $d^1_{CE}\varphi$*-compatible. Therefore,*

$$d^1_*(\varphi) := \left(d^1_{CE}\varphi, ind^1(\varphi)\right) \in C^2_*(L;M).$$

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Definition of the map $d_*^1 : C_*^1(L, M) \longrightarrow C_*^2(L, M)$.

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2. *We have $d^1_* \circ d^0_* = 0$.*

3. *The space $H^1_*(L; M) := Ker\left(d^1_*\right) / Im\left(d^0_*\right)$ is well defined.*

Restricted cohomology for restricted Lie superalgebras

Definition of the map $d_*^2 : C_*^2(L,M) \longrightarrow C_*^3(L,M)$.

An element $(\varphi,\omega) \in C_*^2(L;M)$ induces a map $\mathrm{ind}^2(\varphi,\omega) : L \times L_{\bar{0}} \to M$ defined by

$$\begin{aligned}\mathrm{ind}^2(\varphi,\omega)(x,y) =& \varphi\left(x, y^{[p]}\right) - \sum_{i+j=p-1} (-1)^i y^i \varphi\Big([[\cdots[x,\overbrace{y],\cdots],y}^{j \text{ terms}}], y\Big)\\ &+ (-1)^{|\varphi||x|} x\omega(y),\end{aligned}$$

for $\mathrm{Im}(\omega) \subseteq M_{|\varphi|}$, and then extended using (1).

Restricted cohomology for restricted Lie superalgebras

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Theorem (Bouarroudj-E.)

1. *The map $ind^2(\varphi, \omega)$ is $d_{CE}^2\varphi$-compatible.*
2. *We have $d_*^2 \circ d_*^1 = 0$, where $d_*^2(\varphi, \omega) := \left(d_{CE}^2\varphi, ind^2(\varphi, \omega)\right)$.*

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Definition of the map $d^2_* : C^2_*(L, M) \longrightarrow C^3_*(L, M)$.

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3. *The space $H^2_*(L; M) := Ker\left(d^2_*\right) / Im\left(d^1_*\right)$ is well defined.*

An example. Consider the Lie superalgebra

$$L =< e_1|e_2, e_3 >, \ [e_1, e_2] = e_3, \ e_1^{[p]} = 0.$$

Let $\varphi \in C^2_{\mathrm{CE}}(L; L)$. A map $\omega : L_{\bar{0}} \to L$ is φ-compatible if and only if

$$\omega(\lambda x) = \lambda^p \omega(x) \text{ and } \omega(x + y) = \omega(x) + \omega(y), \ \forall x, y \in L_{\bar{0}}, \ \forall \lambda \in \mathbb{K}. \quad (2)$$

Example of computation

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Lemma

A basis for the Chevalley-Eilenberg 2-cocycles space $Z^2_{CE}(L; L)$ is given by

$$\begin{aligned}
\varphi_1 &= e_1 \otimes \Delta_{1,2} + 2e_2 \otimes \Delta_{2,2}; & \varphi_2 &= -2e_1 \otimes \Delta_{1,3} + 2e_3 \otimes \Delta_{3,3}; \\
\varphi_3 &= e_2 \otimes \Delta_{1,2}; & \varphi_4 &= e_2 \otimes \Delta_{1,3}; \\
\varphi_5 &= 2e_2 \otimes \Delta_{2,2} + e_3 \otimes \Delta_{2,3}; & \varphi_6 &= e_3 \otimes \Delta_{1,2}; \\
\varphi_7 &= e_3 \otimes \Delta_{1,3}; & \varphi_8 &= e_3 \otimes \Delta_{2,2},
\end{aligned}$$

where $\Delta_{i,j}(e_k, e_l) = \delta_{i,k}\delta_{j,l}$ and $\Delta_{i,j} = -(-1)^{|e_i||e_j|}\Delta_{j,i}$.

Example of computation ($p > 3$)

The case where $p > 3$. Let $(\varphi, \omega) \in C^2_*(L; L)$.

Then,

1. $(\varphi, \omega) \in Z^2_*(L; L)$ if and only if $\varphi \in Z^2_{\mathrm{CE}}(L; L)$ and $\omega(e_1) = \gamma e_3$, $\gamma \in \mathbb{K}$;

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3. $\varphi_5 \cong \varphi_1$ and $\varphi_7 \cong \varphi_3$;
4. $\mathrm{ind}^1(\psi) = 0, \ \forall \psi \in C^1_*(L; L)$.

Therefore, we have

$$H^2_*(L; L) = \mathrm{Span}\{(\varphi_1, 0);\ (\varphi_2, 0);\ (\varphi_3, 0);\ (\varphi_4, 0);\ (0, \omega_5)\},$$

where $\omega_5(e_1) = e_3$.

Example of computation ($p = 3$)

The case where $p = 3$. Let $(\varphi, \omega) \in C^2_*(L; L)$. Suppose that

$$\omega(e_1) = \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3,\ \gamma_1, \gamma_2, \gamma_3 \in \mathbb{K}\,.$$

Then,

1. $\forall \varphi \in Z^2_{\mathrm{CE}}(L; L)$, we have $\mathrm{ind}^2(\varphi, \omega)(e_1, e_1) = \gamma_2 e_3$.

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Then,

1. $\forall \varphi \in Z^2_{\mathrm{CE}}(L; L)$, we have $\mathrm{ind}^2(\varphi, \omega)(e_1, e_1) = \gamma_2 e_3$.
2. For $i \neq 4$, we have $\mathrm{ind}^2(\varphi_i, \omega)(e_2, e_1) = \gamma_1 e_3$.
3. For $i = 4$, $\mathrm{ind}^2(\varphi_4, \omega)(e_2, e_1) = (1 - \gamma_1)e_3$.

Therefore,

$$H^2_*(L; L) = \mathrm{Span}\big\{(\varphi_1, 0);\ (\varphi_2, 0);\ (\varphi_3, 0);\ (\varphi_4, \omega_4);\ (0, \omega_5)\big\},$$

where $\omega_4(e_1) = e_1$ and $\omega_5(e_1) = e_3$.

A subcomplex

Let L be a restricted Lie superalgebra and M a restricted L-module. We define a subspace $C^2_*(L;M)^+ \subset C^2_*(L;M)$ by

$$C^2_*(L;M)^+ := \Big\{(\varphi,\omega) \in C^2_*(L;M),\ \mathrm{Im}(\omega) \subseteq M_{\bar{0}}\Big\}.$$

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Lemma

(i) *We have an inclusion* $B^2_*(L;M)_{\bar{0}} \subset C^2_*(L;M)^+$.

(ii) *The space* $C^2_*(L;M)^+$ *is* $\mathbb{Z}_2$*-graded and the degree of an homogeneous element* $(\varphi,\omega) \in C^2_*(L;M)^+$ *is given by* $|(\varphi,\omega)| = |\varphi|$.

This Lemma allows us to consider the space $Z^2_*(L;M)^+ := \ker\big(d^2_{*|C^2_*(L;M)^+}\big)$. Thus we can define

$$H^2_*(L;M)^+ := Z^2_*(L;M)^+/B^2_*(L;M)_{\bar{0}}.$$

The space $H^2_*(L;M)^+$ is $\mathbb{Z}_2$-graded.

Central extensions of restricted Lie superalgebras

Let $\left(L, [\cdot,\cdot], (\cdot)^{[p]}\right)$ be a restricted Lie superalgebra, and M be a strongly abelian restricted Lie superalgebra ($i.e$, $[m,n]=0\ \forall m,n\in M,$ and $m^{[p]}=0\ \forall m\in M_{\bar{0}}$).

A **restricted extension** of L by M is a short exact sequence of restricted Lie superalgebras

$$0 \longrightarrow M \xrightarrow{\iota} E \xrightarrow{\pi} L \longrightarrow 0.$$

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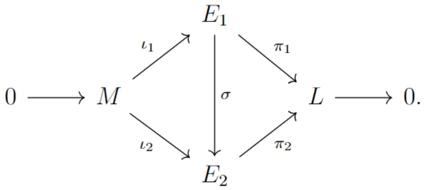
In the case where $\iota(M) \subset \mathfrak{z}(E) := \{a \in E,\ [a, b] = 0\ \forall b \in E\}$, M is a trivial L-module. These extensions are called **restricted central extensions**.

Two restricted central extensions of L by M are called **equivalent** if there is a restricted Lie superalgebras morphism $\sigma : E_1 \to E_2$ such that the following diagram commutes:



Central extensions of restricted Lie superalgebras

$$0 \longrightarrow M \xrightarrow{\iota} E \xrightarrow{\pi} L \longrightarrow 0.$$

Theorem (Bouarroudj-E.)

Let L be a restricted Lie superalgebra and M a strongly abelian restricted Lie superalgebra. Then, the equivalence classes of restricted central extensions of L by M are classified by $H^2_*(L; M)^+_{\bar{0}}$.

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Structure maps on E. Let $(\varphi, \omega) \in Z^2_*(L; M)^+_{\bar{0}}$. The bracket and the p-maps on E are given by

$$[x + m, y + n]_E := [x, y] + \varphi(x, y), \qquad \forall x, y \in L, \ \forall m, n \in M; \tag{3}$$

$$(x + m)^{[p]_E} := x^{[p]} + \omega(x), \qquad \forall x \in L_{\bar{0}}, \ \forall m \in M_{\bar{0}}. \tag{4}$$

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Proposition

Let L be a p-nilpotent restricted Lie superalgebra of dimension n. Then, L is isomorphic to a central extension by a restricted 2-cocycle of a p-nilpotent restricted Lie superalgebra of dimension $n-1$.

Dimension 3

- sdim$(L) = (1|2)$: $L = \langle e_1 | e_2, e_3 \rangle$.

  1. $\mathbf{L}^{1}_{1|2} = \langle e_1 | e_2, e_3 \rangle$ (abelian):
     1. $e_1^{[p]} = 0$;
  2. $\mathbf{L}^{2}_{1|2} = \langle e_1 | e_2, e_3; [e_2, e_3] = e_1 \rangle$:
     1. $e_1^{[p]} = 0$;
  3. $\mathbf{L}^{3}_{1|2} = \langle e_1 | e_2, e_3; [e_1, e_2] = e_3 \rangle$:
     1. $e_1^{[p]} = 0$.
  4. $\mathbf{L}^{4}_{1|2} = \langle e_1 | e_2, e_3; [e_3, e_3] = e_1 \rangle$:
     1. $e_1^{[p]} = 0$;

- sdim$(L) = (2|1)$: $L = \langle e_1, e_2 | e_3 \rangle$.

  1. $\mathbf{L}^{1}_{2|1} = \langle e_1, e_2 | e_3 \rangle$ (abelian):
     1. $e_1^{[p]} = e_2^{[p]} = 0$;
     2. $e_1^{[p]} = e_2, \ e_2^{[p]} = 0$.
  2. $\mathbf{L}^{2}_{2|1} = \langle e_1, e_2 | e_3; [e_3, e_3] = e_2 \rangle$:
     1. $e_1^{[p]} = e_2^{[p]} = 0$;
     2. $e_1^{[p]} = e_2, \ e_2^{[p]} = 0$.

- sdim$(L) = (3|0)$: $L = \langle e_1, e_2, e_3 \rangle$, (see Schneider-Usefi).

  1. $\mathbf{L}^{1}_{3|0} = \langle e_1, e_2, e_3 \rangle$ (abelian):
     1. $e_1^{[p]} = e_2^{[p]} = e_3^{[p]} = 0$;
     2. $e_1^{[p]} = e_2, \ e_2^{[p]} = e_3^{[p]} = 0$;
     3. $e_1^{[p]} = e_2, \ e_2^{[p]} = e_3, \ e_3^{[p]} = 0$.
  2. $\mathbf{L}^{2}_{3|0} = \langle e_1, e_2, e_3; [e_1, e_2] = e_3 \rangle$
     1. $e_1^{[p]} = e_2^{[p]} = e_3^{[p]} = 0$;
     2. $e_1^{[p]} = e_3, \ e_2^{[p]} = e_3^{[p]} = 0$.

The classification method

1. For each 3-dimensional Lie superalgebra of the previous list, we compute the equivalence classes of non-trivial *ordinary* 2-cocycles under the action by automorphisms given by

$$(A\varphi)(x, y) = \varphi\big(A(x), A(y)\big), \ \forall x, y \in L \qquad (5)$$

2. We build the corresponding central extensions.

3. Some of the superalgebras obtained are isomorphic. We detect and remove redundancies.

4. Using Jacobson's Theorem, we check whether the p-maps on the even part are compatible with the odd part.

Dimension 4: the classification. Lie superalgebras.

Theorem

The classification of 4-dimensional nilpotent Lie superalgebras over an algebraically closed field of characteristic different from 2 is given by:

$sdim(L) = (0|4)$: $L = \langle 0|x_1, x_2, x_3, x_4\rangle$

$\mathbf{L^1_{0|4}}$: $[\cdot,\cdot] = 0$.

$sdim(L) = (1|3)$: $L = \langle x_1|x_2, x_3, x_4\rangle$

$\mathbf{L^1_{1|3}}$: *abelian;*

$\mathbf{L^2_{1|3}}$: $[x_1, x_3] = x_4$;

$\mathbf{L^3_{1|3}}$: $[x_2, x_3] = x_1$;

$\mathbf{L^4_{1|3}}$: $[x_1, x_2] = x_3,\ [x_1, x_3] = x_4$;

$\mathbf{L^5_{1|3}}$: $[x_3, x_3] = x_1$;

$\mathbf{L^6_{1|3}}$: $[x_2, x_2] = x_1,\ [x_3, x_4] = x_1$.

$sdim(L) = (2|2)$: $L = \langle x_1, x_2|x_3, x_4\rangle$

$\mathbf{L^1_{2|2}}$: *abelian;*

$\mathbf{L^2_{2|2}}$: $[x_3, x_4] = x_2$;

$\mathbf{L^3_{2|2}}$: $[x_3, x_3] = x_2,\ [x_3, x_4] = x_1$;

$\mathbf{L^4_{2|2}}$: $[x_3, x_3] = [x_4, x_4] = x_2,\ [x_3, x_4] = x_1$;

$\mathbf{L^5_{2|2}}$: $[x_1, x_3] = x_4$;

$\mathbf{L^6_{2|2}}$: $[x_1, x_3] = x_4,\ [x_3, x_3] = x_2$.

$\mathbf{L^7_{2|2}}$: $[x_4, x_4] = x_1$.

$sdim(L) = (3|1)$: $L = \langle x_1, x_2, x_3|x_4\rangle$

$\mathbf{L^1_{3|1}}$: *abelian;*

$\mathbf{L^2_{3|1}}$: $[x_1, x_2] = x_3$;

$\mathbf{L^3_{3|1}}$: $[x_2, x_2] = x_3$;

$\mathbf{L^4_{3|1}}$: $[x_1, x_2] = [x_3, x_4] = x_3$.

$sdim(L) = (4|0)$: $L = \langle x_1, x_2, x_3, x_4|0\rangle$

$\mathbf{L^1_{4|0}}$: *abelian;*

$\mathbf{L^2_{4|0}}$: $[x_1, x_2] = x_3$;

$\mathbf{L^3_{4|0}}$: $[x_1, x_2] = x_3,\ [x_1, x_3] = x_4$.

Dimension 4: the classification. $p|2p$ maps.

Theorem

The p-nilpotent structures on nilpotent Lie superalgebras of total dimension 4 *with* $\dim(L_{\bar{1}}) > 0$ *are given by:*

- $sdim(L) = (0|4)$: *none.*
- $sdim(L) = (1|3)$: $x_1^{[p]} = 0$.
- $sdim(L) = (2|2)$:
  - $x_1^{[p]_1} = x_2^{[p]_1} = 0$;
  - $x_1^{[p]_2} = x_2,\ x_2^{[p]_2} = 0$.
- $sdim(L) = (3|1)$:
  - *Case* $L_{\bar{0}}$ *abelian:*
    - $x_1^{[p]_1} = x_2^{[p]_1} = x_3^{[p]_1} = 0$;
    - $x_1^{[p]_2} = x_2,\ x_2^{[p]_2} = x_3^{[p]_2} = 0$.
    - $x_1^{[p]_3} = x_2,\ x_2^{[p]_3} = x_3,\ x_3^{[p]_3} = 0$.
  - *Case* $L_{\bar{0}} \cong \mathbf{L}^2_{3|0} = \langle x_1, x_2, x_3; [x_1, x_2] = x_3 \rangle$:
    - $x_1^{[p]_4} = x_2^{[p]_4} = x_3^{[p]_4} = 0$;
    - $x_1^{[p]_5} = x_3,\ x_2^{[p]_5} = x_3^{[p]_5} = 0$.

Thank you for your attention!

Main reference:

S. Bouarroudj, Q. Ehret, *Central extensions of restricted Lie superalgebras and classification of p-nilpotent Lie superalgebras in dimension 4*, arXiv:2401.08313.
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